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A theory of stochastic choice under uncertainty

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ABSTRACT

In this paper we propose a characterization of stochastic choice under risk and under uncertainty. We presume that decision makers' actual choices are governed by randomly selected states of mind, and study the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind. The connections of this work to the literatures on random choice, choice behavior when preference are incomplete; choice of menus; and grades of indecisiveness are also discussed.

1. Introduction

In this paper, we develop a theory of random choice under uncertainty and under risk motivated by the recognition that there are situations in which the decision maker's tastes are subject to random variations. In these situations, a decision maker's choice behavior displays a stochastic pattern represented by a probability distribution on the set of alternatives.

The idea advanced in this paper is that variability in choice behavior is an expression of internal conflict among distinct inclinations, or distinct "selves" of the decision maker, whose assessments of the alternatives are different. We refer to these inclinations as "states of minds" and assume that, analogous to a state of nature, a state of mind resolves the uncertainty surrounding a decision maker's true subjective beliefs and/or tastes. Our theory presumes that, at a meta level, decision makers entertain beliefs about their likely state of mind when having to choose among uncertain, or risky, prospects; that their actual choice is determined by the state of mind that obtains; and that the observed choice probabilities are consistent with these beliefs. In other words, a decision maker's state of mind governs his choice behavior in the sense that, when having to choose among acts (or lotteries), a state of mind, encompassing beliefs and risk attitudes, is selected at random and that state of mind determines which alternative is chosen. The focus of our investigation is *the representation of the decision maker's perception of the stochastic process underlying the selection of his state of mind*. We presume that this process is accessible by introspection and that it agrees with the empirical distribution characterizing the random choice rule.

The fact that states of mind are preference relations has two crucial implications: It renders the evaluation of the outcomes – acts or lotteries, as the case may be – dependent on the state (of mind) and it lends the states of mind the inherently quality of private information (as opposed to states of nature which are observable). These implications raise two difficulties. First, because the preference relation is state dependent, subjective expected utility theory fails to deliver a unique prior. Second, because states of mind are private information, they express themselves, indirectly, through choices among menus rather than directly through the choice of acts. To overcome the first difficulty, we apply a modified version of the model of Karni and Schmeidler (1980, 2016). To overcome the second difficulty, building on ideas introduced by Kreps (1979) and developed by Dekel et al. (2001), we derive preferences over acts from those on menus. Hence, we assume that a decision maker is characterized by two primitive preference relations: a preference relation on the set of menus of alternatives depicting his actual choice behavior and an introspective preference relation on hypothetical mental state-act lotteries.

The preference relation on the set of menus induces preferences on the set of mental acts (that is, mappings from the set of

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[1] Part of this work was done during my visit to EIEF, Rome.

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states of mind to the set of uncertain, or risky, prospects). Both the preference relation on the set of mental acts and that on the mental state-act lotteries are assumed to satisfy the von Neumann–Morgenstern axioms and, when a natural correspondence connects between their domains, they are required to agree with each other. This model yields a representation of the preference relations over mental acts induced by menus that takes the form of subjective expected utility with state-dependent utility functions defined on uncertain, or risky, prospects and a unique subjective prior on the set of states of mind. The distribution on the mental state space characterizes the decision maker's stochastic choice behavior.

More formally, let $\{\succcurlyeq_\omega \mid \omega \in \Omega\}$ be a set of preference relations on the set, H, of Anscombe and Aumann (1963) acts, and assume that they satisfy the axioms of expected utility theory. A menu, M, is a non-empty compact subset of Anscombe–Aumann acts. An act induced by M, denoted f_M, is an assignment to each $\omega \in \Omega$ of an act $h \in M$ such that $h \succcurlyeq_\omega h'$, for all $h' \in M$. We denote by F the set of acts induced by menus. Let $\hat{\succcurlyeq}$ be a preference relation on the set of all menus. Define the induced preference relation on F as follows: $f_M \succcurlyeq f_{M'}$ if $M \hat{\succcurlyeq} M'$. Broadly speaking, the main result of this paper is identifying necessary and sufficient conditions that yield the following representation: There exist a continuous, non-constant, real-valued function u on $\Omega \times H$ that is affine in its second argument and is unique up to positive linear transformation, and an essentially unique probability distribution η on Ω such that, for all $f_M, f_{M'} \in F$,

$$f_M \succcurlyeq f_{M'} \Leftrightarrow \sum_{\omega\in\Omega} \eta(\omega)\left[u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega))\right] \geq 0.$$

Moreover, for every two acts h and h', the probability of choosing h over h' is given by

$$\Pr\left(h \mid \left\{h, h'\right\}\right) = \eta\left(\left\{\omega \in \Omega \mid u(\omega, h) > u\left(\omega, h'\right)\right\}\right).$$

In the context of risk, this representation is similar to that of Dekel et al. (2001). However, the uniqueness of η is specific to our model.[2]

The theory developed in this paper is related not only to the literature on random choice but also to the literature on choice behavior when preference relations are incomplete, the literature on choice of menus, and the work on grades of indecisiveness.

Applying our model to menus of lotteries, we show that our theory implies the axioms of Gul and Pesendorfer (2006). Hence, the probability measure η generates their random utility and random choice model. All our preference relations are defined ex ante, at an earlier stage, before the actual choice among various acts/lotteries. In that stage, the decision maker chooses among menus of alternatives. We do not make explicit the later, ex post, choice, but, as indicated above, we assume that it is consistent with the expectations the decision maker has at the earlier stage. To make the connection between the two stages more explicit, one can follow Ahn and Sarver (2013), who join together the ex ante model of Dekel et al. (2001) with the ex post random choice of Gul and Pesendorfer (2006).

The representation of incomplete preferences under uncertainty specifies a set of probability–utility pairs and requires that one alternative be strictly preferred over another if and only if the former yields higher subjective expected utility than the latter according to each probability–utility in the set.[3] In this context, we identify states of mind with probability–utility pairs. When the alternatives are noncomparable, the choice may be random. Our model implies that the likelihood that one alternative is chosen over another is the measure (according to η) of the subset of the states of mind that prefer that alternative.

We also show that Minardi and Savochkin's (2015) notion of grades of indecisiveness between two Anscombe–Aumann acts, say f and g, can be represented by the probability η of the set $\{\omega \in \Omega \mid f \succ_\omega g\}$.

The model developed in this paper is related to the literature on probabilistic choice originated by Luce and Suppes (1965) and later developed by Loomes and Sugden (1995). Recently, Melkonyan and Safra (2016) axiomatized the utility components of two families of such preferences, where one family satisfies the independence axiom. Our paper complements and extends that model by characterizing the inherent probability distribution over the possible states of mind (possible tastes).

A more detailed discussion of the connections between this paper and these branches of the literature appears in Section 3, following the presentation of our theory in the next section. The proofs are relegated to Section 4.

2. Stochastic choice theory

2.1. *The analytical framework: revealed preferences over mental acts induced by menus*

2.1.1. *Acts and preferences*

Let X be a finite set of *outcomes*, and denote by $\Delta(X)$ the set of all probability measures on X. For each $p, q \in \Delta(X)$, and $\alpha \in [0, 1]$, define $\alpha p + (1-\alpha) q \in \Delta(X)$ by $(\alpha p + (1-\alpha) q)(x) = \alpha p(x) + (1-\alpha) q(x)$, for all $x \in X$.

Let S be a finite set of *material states* (or states of nature), and denote by H the set of all mappings from S to $\Delta(X)$. Elements of H are referred to as *acts*.[4] For all $h, h' \in H$, and $\alpha \in [0, 1]$, define $\alpha h + (1-\alpha) h' \in H$ by $\left(\alpha h + (1-\alpha) h'\right)(s) = \alpha h(s) + (1-\alpha) h'(s)$, for all $s \in S$, where the convex operation $\alpha h(s) + (1-\alpha) h'(s)$ is defined as above. Under this definition, H is a convex subset of the linear space $\mathbb{R}^{|X|\cdot|S|}$.

Let $\mathcal{P}$ be the set of all preference relations on H whose structure is depicted by the following axioms:

(A.1) (**Strict total order**) The preference relation $\succ$ is asymmetric and negatively transitive.
(A.2) (**Archimedean**) For all $h, h', h'' \in H$, if $h \succ h'$ and $h' \succ h''$, then $\beta h + (1-\beta) h'' \succ h'$ and $h' \succ \alpha h' (1-\alpha) h''$ for some $\alpha, \beta \in (0, 1)$.
(A.3) (**Independence**) For all $h, h', h'' \in H$ and $\alpha \in (0, 1]$, $h \succ h'$ if and only if $\alpha h + (1-\alpha) h'' \succ \alpha h' + (1-\alpha) h''$.
(A.4) (**Nontriviality**) $\succ$ is not empty.

By the expected utility theorem, a preference relation satisfies (A.1)–(A.4) if and only if there exists a nonconstant real-valued function, $w(x, s)$, on $X \times S$, unique up to cardinal unit-comparable transformation,[5] such that, for all $h, h' \in H$,[6]

$$h \succ h' \Leftrightarrow \sum_{s\in S}\sum_{x\in X} w(x, s)\left[h(s)(x) - h'(s)(x)\right] > 0.$$

[2] Sadowski (2013) obtained uniqueness of the probabilities in the model of Dekel et al. (2001) by enriching the model with objective states.

[3] See Galaabaatar and Karni (2013). In the case of incomplete preferences under risk, we identify states of mind with utility function and the analogous results are Dubra et al. (2004) and Shapley and Baucells (2008).

[4] See Anscombe and Aumann (1963).

[5] A function $\hat{w}(x, s)$ is said to be cardinal unit-comparable transformation of $w(x, s)$ if there exist a real number $b > 0$ and $a \in \mathbb{R}^S$ such that $\hat{w}(x, s) = bw(x, s) + a(s)$, for all $(x, s) \in X \times S$.

[6] See Kreps (1988).

2.1.2. States of mind and mental acts induced by menus

Let Ω be a finite, nonempty set and consider the subset of preferences $\mathcal{P}^{\Omega} = \{\succ_\omega \in \mathcal{P} \mid \omega \in \Omega\}$. We refer to $\succ_\omega$ as a *state of mind* depicting a possible mood, or persona, of the decision maker. To avoid notational redundancy we assume that $\succ_\omega \neq \succ_{\omega'}$ for all $\omega \neq \omega'$. To simplify the notation, we also identify $\succ_\omega$ with ω and refer to Ω as the *mental state space*. Let $\succcurlyeq_\omega$ be a binary relation on H defined by $h \succcurlyeq_\omega h'$ if $\neg\left(h' \succ_\omega h\right)$. Then, $\succcurlyeq_\omega$ is complete and transitive.

A *menu* is a nonempty compact subset of H. Let $\mathcal{M}$ be the set of menus, and denote by M its generic element. For each $M \in \mathcal{M}$, define a correspondence $\varphi_M : \Omega \rightrightarrows H$ as follows: For every $\omega \in \Omega$,

$$\varphi_M(\omega) = \{h \in M \mid h \succcurlyeq_\omega h', \ \forall h' \in M\}.$$

The correspondence φ_M maps mental states to subsets of ω-equivalent acts in H.

Let $\hat{F} := \{f : \Omega \to H\}$ be the set of all mappings from Ω to H. Elements of $\hat{F}$ are referred to as *mental acts*. The set $\hat{F}$ is a convex set (with respect to the operation $\left(\alpha f + (1-\alpha) f'\right)(\omega) = \alpha f(\omega) + (1-\alpha) f'(\omega)$, for all $\omega \in \Omega$). Let $F := \{f \in \hat{F} \mid \exists M \in \mathcal{M} : \forall \omega, \ f(\omega) \in \varphi_M(\omega)\}$. An element of F is dubbed a *mental act induced by M*. It is an element of $\hat{F}$ that maximizes all preference relations $\omega \in \Omega$ over M. We denote by $f_M \subset F$ the set of *mental acts induced by M*. Let $\hat{\succcurlyeq}$ be a preference relation on $\mathcal{M}$ depicting the decision maker's observable behavior when faced with choice among menus. We assume that $\hat{\succcurlyeq}$ satisfies the analogue of axioms (A.1)–(A.4), where mixtures of menus are defined by mixing all possible pairs of acts in which the acts belong to distinct menus. Define an induced preference relation $\succ$ on F by $f \succ f'$ if $f \in f_M$, $f' \in f_{M'}$ and $M \hat{\succcurlyeq} M'$. This definition presumes that, when decision makers compare two menus, M and M', they imagine selecting mental acts from f_M and $f_{M'}$, respectively, and comparing them. Clearly, if $f, f' \in f_M$ then $f \sim f'$. Henceforth, we treat f_M as an element of F. Unlike $\hat{\succcurlyeq}$, the induced preference relation $\succ$ on F is defined on mental constructs which are not directly observable and must be inferred from the decision maker's choice of menus. Clearly, the induced preference relation $\succ$ on F satisfies the axioms (A.1)–(A.4). The following lemma ensures that these requirements are nonvacuous.

Lemma 1. *F is a convex set.*

Remark 1. The mental state space is analogous to the subjective state space introduced by Kreps (1979). However, unlike Kreps, who derived the existence of an infinite subjective state space from preference for flexibility (that is, from preferences over menus), we take the existence of a finite subjective state space as a primitive aspect of the model (which finds its expression in preferences over menus). To construct the subjective state space, we can use the approach of Karni (2015) according to which there is a finite set, $\tilde{F}$, of alternatives. Menus are nonempty subsets of $\tilde{F}$. Let $\tilde{\mathcal{M}}$ denote the set of all menus consisting of elements of $\tilde{F}$. The subjective state space induced by $\tilde{\mathcal{M}}$ is the set of mappings $\tilde{\Omega} := \{\omega : \tilde{\mathcal{M}} \to \tilde{F} \mid \omega(M) \in M, \forall M \in \tilde{\mathcal{M}}\}$. Note that, if alternatives are agreed upon by distinct observers, then the derived state space is objective and is determined independently of the preferences of the decision maker. As pointed out by Karni (2015), in general, the states in $\tilde{\Omega}$ do not correspond to complete and transitive preference relations. However, suppose that there is a choice function $c : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ (i.e., $c(M) \subseteq M$, for all $M \in \tilde{\mathcal{M}}$) that satisfies the weak axiom of revealed preference. If for all $\omega, \omega(M) \in c(M)$ for all M, then each state correspond to a complete and transitive preference relation, $\succcurlyeq_\omega$ on $\tilde{F}$. Moreover, the aforementioned state space is unique.

In some applications (e.g., when the preference relation on acts, in the case of uncertainty, or lotteries, in the case of risk, is incomplete) the existence and uniqueness are implied by the representation (see discussion in Section 3.1).

2.2. The analytical framework: state-act lotteries and introspective preferences

2.2.1. State-act lotteries and their decomposition

Let $\hat{L}(\Omega \times H)$ be the set of probability distributions on $\Omega \times H$ with finite supports. Elements of $\hat{L}(\Omega \times H)$ are *state-act lotteries*. A state-act lottery $\ell \in \hat{L}(\Omega \times H)$ is said to be *non-degenerate* if $\mu_\ell(\omega) := \Sigma_{h \in H} \ell(\omega, h) > 0$, for every $\omega \in \Omega$. Clearly, $\mu_\ell \in \Delta(\Omega)$.

Assume that ℓ is non-degenerate and consider the function $J : \hat{L}(\Omega \times H) \to \hat{F}$ defined by

$$J(\ell)(\omega) = \sum_{h \in H} \frac{\ell(\omega, h)}{\mu_\ell(\omega)} h, \quad \text{for all } \omega \in \Omega.$$

For the uniform distribution $\lambda(\omega) = \frac{1}{|\Omega|}$ in $\Delta(\Omega)$, define the function $K : F \to \hat{L}(\Omega \times H)$ by

$$K(f)(\omega, h) = \begin{cases} \dfrac{1}{|\Omega|} & f(\omega) = h \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $K(f)$ is non-degenerate and $J(K(f)) = f$. Henceforth, we focus the attention on $\ell \in \hat{L}(\Omega \times H)$ such that $J(\ell) \in F$. Define

$$L(\Omega \times H) = \{\ell \in \hat{L}(\Omega \times H) \mid \ell \text{ non-degenerate and } J(\ell) \in F\}.$$

Lemma 2. *$L(\Omega \times H)$ is a convex set.*

2.2.2. Introspective preferences and consistency

Let $\succ^*$ be a binary relation on $L(\Omega \times H)$ and assume that $\succ^*$ satisfies the analogue of axioms (A.1)–(A.4).[7] We refer to $\succ^*$ as *introspective preference relation*. Define $\succcurlyeq^*$ on $L(\Omega \times H)$, by $\ell \succcurlyeq^* \ell'$ if $\neg\left(\ell' \succ^* \ell\right)$. These preference relations express the decision maker's beliefs about his behavior if he could choose between lotteries $L(\Omega \times H)$. We presume that the decision maker is conscious that he may experience different moods, and that he is capable of expressing preferences not only among acts given a certain mood (which is captured by the state of mind $\succ_\omega$) but also across acts in different moods. For example, the decision maker is supposed to be able to claim, upon introspection, that he prefers listening to Beethoven 9th symphony when in good mood over listening to Mozart's requiem when depressed. Because, in general, the decision maker does not get to choose his mood, the introspective preferences are hypothetical and can only be expressed verbally.

Two state-act lotteries ℓ and ℓ' are said to *agree outside ω* if $\ell\left(\omega', \cdot\right) = \ell'\left(\omega', \cdot\right)$, for all $\omega' \in \Omega \setminus \{\omega\}$. Similarly, two mental acts, f and f' are said to *agree outside ω* if $f\left(\omega'\right) = f'\left(\omega'\right)$, for all $\omega' \in \Omega \setminus \{\omega\}$. Following Karni and Schmeidler (2016) we make the following definition and axiom.

Definition 1. A state of mind $\omega \in \Omega$ is *obviously null* if $f \sim f'$ for all $f, f' \in F$ that agree outside ω, and there exist $\ell, \ell' \in L(\Omega \times H)$ that agree outside ω, such that $\ell \succ^* \ell'$; it is *obviously nonnull* if there are $f, f' \in F$ that agree outside ω and $f \succ f'$.

The following example demonstrates that the definitions above are not vacuous.

[7] Recall that in expected utility theory, the set of outcomes that constitutes the support of the lotteries is arbitrary. Consequently, the set of state-act lotteries in this paper is a special case of the theory of von Neumann and Morgenstern, and the applicability of the axioms is implied.

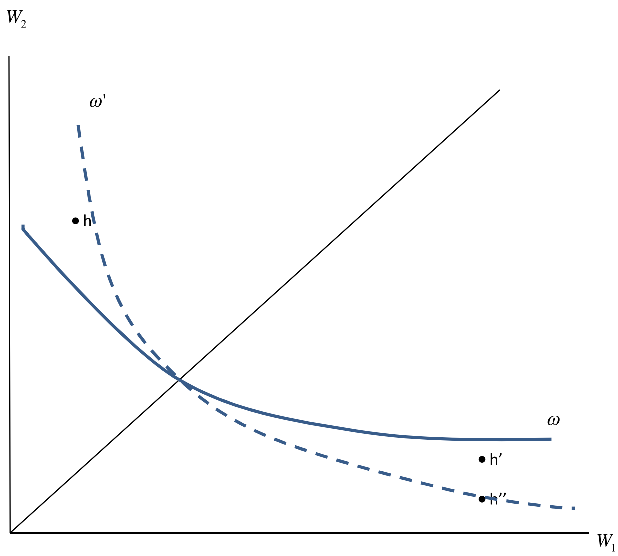

Fig. 1. The first axis corresponds to income in material state s_1, while the second corresponds to income in s_2. The bold indifference curve belongs to the preference relation of mental state ω, while the dashed one belongs to that of mental state ω'.

Example. Assume there are two material states ($S = \{s_1, s_2\}$), let $\Omega = \{\omega, \omega'\}$ and consider $M = \{h, h'\}$ and $M' = \{h, h''\}$ such that $h \succ_\omega h' \succ_\omega h''$, $h' \succ_{\omega'} h'' \succ_{\omega'} h$. Then $f_M(\omega) = h$, $f_M(\omega') = h'$ and $f_{M'}(\omega) = h$, $f_{M'}(\omega') = h''$ (see Fig. 1 where, for simplicity, h, h' and h'' assign degenerate lotteries with both material states s_i). Denote $\ell_M = K(f_M)$ and $\ell_{M'} = K(f_{M'})$. Then $\ell_M(\omega, h) = \frac{1}{2} = \ell_{M'}(\omega, h)$, $\ell_M(\omega, \hat{h}) = 0 = \ell_{M'}(\omega, \hat{h})$ for all $\hat{h} \in H \setminus \{h\}$, $\ell_M(\omega', h') = \frac{1}{2} = \ell_{M'}(\omega', h'')$, $\ell_M(\omega', \hat{h}) = 0$, for all $\hat{h} \in H \setminus \{h'\}$ and $\ell_{M'}(\omega', \hat{h}) = 0$ for all $\hat{h} \in H \setminus \{h''\}$. Hence, ℓ_M agrees with $\ell_{M'}$ outside ω'. Suppose that $\ell_M \succ^* \ell_{M'}$. If $J(\ell_M) \sim J(\ell_{M'})$ then ω' is obviously null and if $J(\ell_M) \succ J(\ell_{M'})$ then ω' is obviously nonnull.

(A.5) **Consistency I.** For all $\omega \in \Omega$ and all non-degenerate $\ell, \ell' \in L(\Omega \times H)$ such that ℓ agrees with ℓ' outside ω: if $J(\ell) \succ J(\ell')$ then $\ell \succ^* \ell'$.

2.3. Representation of preferences over menus and introspective preferences: uncertainty

The following theorem asserts the existence and describes the uniqueness properties of subjective expected utility representations of the preference relations $\succcurlyeq$ on the set of mental acts induced by menus and $\succcurlyeq^*$ on the set of state-act lotteries. This theorem extends theorem 3 of Karni and Schmeidler (1980): whereas their domain is the entire set of state-prize lotteries, ours is the more complex set of mental-state-act lotteries that are mapped into mental acts induced by menus.

Theorem 1. *Let $\succcurlyeq^*$ on $L(\Omega \times H)$ and $\succcurlyeq$ on F be binary relations. The following conditions are equivalent:*

(a.i) *The asymmetric parts of $\succcurlyeq^*$ and $\succcurlyeq$ satisfy* (A.1)–(A.4) *and jointly they satisfy* (A.5).

(a.ii) *There exist continuous, non-constant, real-valued function u on $\Omega \times H$ that is affine in its second argument, and a probability distribution η on Ω such that, for all $f_M, f_{M'} \in F$,*

$$f_M \succcurlyeq f_{M'} \Leftrightarrow \sum_{\omega \in \Omega} \eta(\omega) \left[u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega)) \right] \geq 0, \qquad (1)$$

and, for all $\ell, \ell' \in L(\Omega \times H)$,

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{\omega \in \Omega} \sum_{h \in H} u(\omega, h)\, \ell(\omega, h) \geq \sum_{\omega \in \Omega} \sum_{h \in H} u(\omega, h)\, \ell'(\omega, h). \qquad (2)$$

(b) *u is unique up to positive linear transformation.*

(c) *For obviously null $\omega \in \Omega$, $\eta(\omega) = 0$, and for obviously non-null $\omega \in \Omega$, $\eta(\omega) > 0$. Moreover, if all states of mind are obviously non-null then η is unique.*

The proof is given in Section 5.

Applying the model of Karni and Schmeidler to the preference relations $\succ_\omega \in \mathscr{P}$, it can be shown that the utility function in the representations in Theorem 1 takes the linear form $u(\omega, h) = \Sigma_{s \in S} \pi(\omega, s) \, \Sigma_{x \in X} u(\omega, x)\, h(s)(x)$, where $\pi(\omega, \cdot)$ is a probability measure on the material state space, S, representing the beliefs of a decision maker whose mood is $\succ_\omega$.

Finally, note that, by definition and (1), the representation of $\hat{\succsim}$ on $\mathscr{M}$ is as follows:

$$M \hat{\succsim} M' \Leftrightarrow \sum_{\omega \in \Omega} \eta(\omega) \left[u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega)) \right] > 0.$$

The importance of the preference relation on menus is that, unlike $\succcurlyeq$ on F, which is inferred relation, it is directly observable in the sense of depicting actual choice behavior.

2.4. Representation of preferences over menus and introspective preferences: risk

2.4.1. The analytical framework: mental acts induced by menus

The analysis of preferences over menus under risk is the special case of preferences over menus under uncertainty in which the set of material states space is a singleton and can be ignored. Here, Ω is a finite, nonempty set representing preferences over $\Delta(X)$, menus are non-empty compact subsets of $\Delta(X)$ and $\mathscr{M}_r$ is the set of all menus. For each $M \in \mathscr{M}_r$, define the correspondence $\varphi_M^r : \Omega \rightrightarrows \Delta(X)$ as follows: For every $\omega \in \Omega$,

$$\varphi_M^r(\omega) = \{ p \in M \mid p \succcurlyeq_\omega q, \ \forall q \in M \}.$$

The correspondence φ_M^r maps the set of mental states to subsets of ω-equivalent lotteries. In the present context, one interpretation of mental state is risk attitude.

Let $\hat{G} := \{ g : \Omega \to \Delta(X) \}$ be the set of all mappings from Ω to $\Delta(X)$. Elements of $\hat{G}$ are dubbed *AA mental acts.*[8] Clearly, $\hat{G}$ is a convex set under the usual definition. Let $G := \{ g \in \hat{G} \mid \exists M \in \mathscr{M}_r : \forall \omega, \ g(\omega) \in \varphi_M^r(\omega) \}$. We denote by $g_M \subset G$ the set of *AA mental acts induced by* M. Let $\hat{\succsim}_r$ be a preference relation on $\mathscr{M}_r$ depicting the decision maker's observable behavior when faced with choice among menus. As before, we assume that $\hat{\succsim}_r$ satisfies the analogue of axioms (A.1)–(A.4) where mixtures of menus are defined by mixing all possible pairs of acts in which the acts belong to distinct menus. Define an induced preference relation $\succ_G$ on G by $g \succ g'$ if $g \in g_M$, $g' \in g_{M'}$ and $M \hat{\succsim}_r M'$. By argument analogous to Lemma 1, G is a convex set. We assume that the induced preference relation $\succ_G$ on G satisfies the analogue of axioms (A.1)–(A.4).

2.4.2. The analytical framework: mental states-roulette lotteries

Let $\hat{L}(\Omega \times \Delta(X))$ be the set of simple probability distributions over $\Omega \times \Delta(X)$. As before, let $\mu_\ell(\omega) := \Sigma_{p \in \Delta(X)} \ell(\omega, p)$, for every $\omega \in \Omega$.

8 AA for Anscombe and Aumann.

Assume that ℓ is non-degenerate and consider the function $I : \hat{L}(\Omega \times \Delta(X)) \to \hat{G}$ defined by

$$I(\ell)(\omega) = \sum_{p \in \Delta(X)} \frac{\ell(\omega, p)}{\mu_\ell(\omega)} p, \quad \text{for all } \omega \in \Omega.$$

For the uniform distribution $\mu(\omega) = \frac{1}{|\Omega|}$ in $\Delta(\Omega)$, define the function $T : G \to \hat{L}(\Omega \times \Delta(X))$ by

$$T(g)(\omega, p) = \begin{cases} \frac{1}{|\Omega|} & g(\omega) = p \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $T(g)$ is non-degenerate and $I(T(g)) = g$. Define

$$L(\Omega \times \Delta(X)) = \{\ell \in \hat{L}(\Omega \times \Delta(X)) \mid \ell \text{ non-degenerate and } I(\ell) \in G\}.$$

By the same argument as in the proof of Lemma 2, $L(\Omega \times \Delta(X))$ is a convex set.

2.4.3. Consistency and representation

Let $\succ_L$ be a preference relation on $L(\Omega \times \Delta(X))$ and assume that it satisfies the analogue of (A.1)–(A.4). Analogously to Definition 1, a state of mind $\omega \in \Omega$, is said to be *obviously null* if, for all $g, g' \in G$ such that g agrees with g' outside ω, $g \sim_G g$ and there exist that $\ell, \ell' \in L(\Omega \times \Delta(X))$ such that ℓ agrees with ℓ' outside ω, and $\ell \succ_L \ell'$. It is *obviously nonnull* if $g \succ_G g$, for some $g, g' \in G$ such that g agrees with g' outside ω.

The next axiom is analogous to (A.5).

(A.6) **Consistency II**. For all $\omega \in \Omega$, and all non-degenerate $\ell, \ell' \in L(\Omega \times \Delta(X))$ such that ℓ agrees with ℓ outside ω, $I(\ell) \succ_G I(\ell')$ implies $\ell \succ_L \ell'$.

Corollary. *Let $\succcurlyeq_L$ on $L(\Omega \times \Delta(X))$ and $\succcurlyeq_G$ on G be binary relations then the following conditions are equivalent:*

(a.i) *The asymmetric parts of $\succcurlyeq_L$ and $\succcurlyeq_G$ satisfy* (A.1)–(A.4) *and jointly they satisfy* (A.6).

(a.ii) *There exist a non-constant, real-valued function u on $\Omega \times \Delta(X)$ affine in its second argument, and a probability distributions λ on Ω such that for $g^*_M, g^*_{M'} \in G$,*

$$g^*_M \succcurlyeq_G g^*_{M'} \Leftrightarrow \sum_{\omega \in \Omega} \lambda(\omega) \left[u\left(\omega, g^*_M(\omega)\right) - u\left(\omega, g^*_{M'}(\omega)\right)\right] \geq 0, \quad (3)$$

and, for all $\ell, \ell' \in L(\Omega \times \Delta(X))$,

$$\ell \succcurlyeq_L \ell' \Leftrightarrow \sum_{\omega \in \Omega} \left[\sum_{p \in \Delta(X)} u(\omega, p)\,\ell(\omega, p) - \sum_{p \in \Delta(X)} u(\omega, p)\,\ell'(\omega, p)\right] \geq 0. \quad (4)$$

(b) *u is unique up to positive linear transformation.*

(c) *If $\omega \in \Omega$ is obviously null then $\lambda(\omega) = 0$, and if it is obviously nonnull then $\lambda(\omega) > 0$. Moreover, if all states of mind are obviously non-null then λ is unique.*

The proof is similar to that of Theorem 1 and is not given here.

Remark 2. For every $\omega \in \Omega$, and , $\ell \in L(\Omega \times \Delta(X))$, $\ell(\omega, \cdot)$ is a compound lottery. Hence, by the reduction of compound lottery axiom,

$$\ell(\omega, p)(x) = \sum_{p \in \Delta(X)} \ell(\omega, p)\,p(x), \quad \forall x \in X.$$

Since $u(\omega, \cdot)$ in (4) is affine, $u(\omega, p) = \Sigma_{x \in X} u(\omega, x)\,p(x)$. Hence, for every $\omega \in \Omega$, and $\ell \in L(\Omega \times \Delta(X))$,

$$\sum_{p \in \Delta(X)} u(\omega, p)\,\ell(\omega, p) = \sum_{x \in X} u(\omega, x) \sum_{p \in \Delta(X)} \ell(\omega, p)\,p(x).$$

3. Relation to the literature

3.1. Choice behavior when preferences are incomplete

When the preference relation is complete, there is no distinction between preference and choice behavior. The representation of the preferences is the choice criterion. By contrast, when the preference relation is incomplete, and the choice is between non-comparable alternatives, the representation does not indicate which of the alternatives will be selected. In particular, the choice between alternatives that are non-comparable may be random. This, however, does not mean that non-comparable alternatives are equally likely to be selected. If one alternative is "almost better" then the other, then it stands to reason that it is more likely to be chosen. To lend this idea concrete meaning we note that, in general, in subjective expected utility theory with incomplete preferences one alternative is strictly preferred over another if its subjective expected utility is greater according to a set of pairs of utilities and subjective probabilities.[9] Special cases include complete tastes, in which one alternative is strictly preferred over another if its subjective expected utility is greater according to a set of subjective probabilities, and complete beliefs, in which one alternative is strictly preferred over another if its subjective expected utility is greater according to a set utilities functions. Similarly, in expected utility theory under risk, one alternative is strictly preferred over another if its expected utility is greater according to a set of utilities functions.[10]

Note that if the representation involves a set, Ψ, of probability–utility pairs, as in the case of subjective expected utility theory with incomplete preferences, or a set of utility functions, $\mathcal{U}$, as in the case of expected utility theory under risk with incomplete preferences, then *the set of states of mind is uniquely defined*. More specifically, each $(\pi, U) \in \Psi$ defines a state of mind $\succ_\omega$ which is the preference relation on F induced by the functional $\sum_{s \in S} \pi(s) \sum_{x \in X} U(x) f(x, s)$. Similarly, in the case of risk, each $u \in \mathcal{U}$ defines a state of mind $\succ_\omega$ which is the preference relation on G induced by the functional $\sum_{x \in X} u(x) p(x)$. The uniqueness is an implication of the uniqueness of the corresponding representations. To state the uniqueness result, consider first the case of incomplete preferences under risk. Following Dubra et al. (2004) denote by $\langle \mathcal{U} \rangle$ the closure of the convex cone generated by all the functions in $\mathcal{U}$ and all the constant function on $\Delta(X)$. If $\mathcal{V}$ is another set of utility functions representing the same incomplete preference relation under risk, then $\langle \mathcal{V} \rangle = \langle \mathcal{U} \rangle$.[11] Galaabaatar and Karni (2013) obtained analogous uniqueness result in the case of incomplete preferences under uncertainty.

In all of these instances, two alternatives are non-comparable if one is preferred over the other according to some elements in the corresponding set (e.g., some probability–utility pairs) and the second alternative is preferred over the first according to the rest of the elements in the corresponding set. It seems natural to suppose that the likelihood that the first alternative is chosen depends on the measure of the set of utilities and/or probabilities, as the case may be, according to which the expected utility of the first is larger than that of the second. This presumption expresses the idea that one probability–utility pair is selected at random and the corresponding state of mind governs the particular choice. The

9 See Galaabaatar and Karni (2013).

10 See Bewley (1986), Galaabaatar and Karni (2013), Dubra et al. (2004), Shapley and Baucells (2008) and Galaabaatar and Karni (2012).

11 Dubra et al. (2004) includes that case in which X is a compact set in a metric space. The result for the case in which X is finite appears in by Galaabaatar and Karni (2012).

question is what is the appropriate measure on the mental state space that describes this random selection process?

The theory developed in this paper suggests that the set of probability–utility pairs correspond to the set, Ω, of states of mind; that the decision maker can assess the likelihoods of distinct states of mind by introspection; and that the likelihood of a particular state of mind (or utility probability pair) is selected to decide between the alternatives is given by η.

Suppose that, when facing a choice among acts, the decision maker behaves as if a state of mind from Ω is drawn according to the distribution η, and that this state of mind determines his choice. Specifically, given a menu $M = \{f_1, \ldots, f_n\}$, and assuming that all elements of $\varphi_M(\omega)$ are selected with equal probabilities, the probability that f_i is chosen is α_i, $i = 1, \ldots, n$, is as follows: Let $\Psi_M(f_i) := \{\omega \in \Omega \mid f_i \in \varphi_M(\omega)\}$

$$\alpha_i = \sum_{\omega \in \Psi_M(f_i)} \eta(\omega) \frac{1}{|\varphi_M(\omega)|}. \tag{5}$$

A special case concerns doubleton menus. Let $M = \{f, g\}$ and denote by $\zeta(f, g)$ the probability that f is chosen from the menu M. According to our approach,

$$\zeta(f, g) = \sum_{\omega \in \Psi_M(f)} \eta(\omega) \frac{1}{|\varphi_M(\omega)|}. \tag{6}$$

Consider next the case of subjective expected utility theory with incomplete preferences. Galaabaatar and Karni (2013) define a weak preference relation $\succcurlyeq_{GK}$ on H as follows: For all $f, g \in H$, $f \succcurlyeq_{GK} g$ if $h \succ f$ implies $h \succ g$ for all $h \in H$. Thus, $f \succcurlyeq_{GK} g$ if and only if there exists a set, Ψ, of affine utility functions on $\Delta(X)$ and probability measures on S such that $\sum_{s\in S} \hat{\pi}(s) \sum_{x\in X} \hat{U}(x)[f(x,s) - g(x,s)] = 0$ for some $\left(\hat{\pi}, \hat{U}\right) \in \Psi$ and $\sum_{s\in S} \pi(s) \sum_{x\in X} U(x)[f(x,s) - g(x,s)] > 0$ holds for all $(\pi, U) \in \Psi \setminus \left\{\left(\hat{\pi}, \hat{U}\right)\right\}$.

Consider the case in which Ψ is finite. Applying the theory of this paper we obtain the following implications: (a) $f \succ g$ if and only if $\zeta(f, g) = 1$ (that is, $f \succ_\omega g$ for every $\omega \in \Omega$). (b) $f \succcurlyeq_{GK} g$ if and only if $f \succcurlyeq_\omega g$ for every $\omega \in \Omega$, with indifference for a subset $\hat{\Omega}$ of Ω. The probability that f is selected is $\zeta(f, g) = 1 - \eta\left(\hat{\Omega}\right)/2$. (c) If f and g are non-comparable (that is, $\neg(f \succcurlyeq_{GK} g)$ and $\neg(g \succcurlyeq_{GK} f)$) then the probability that f is selected over g is $\zeta(f, g) \in (0, 1)$.

3.2. Grades of indecisiveness

In a recent paper, Minardi and Savochkin (2015) address the issue of choice in the context of incomplete beliefs, represented by a set of priors. They model a decision maker's inclination to choose one Anscombe–Aumann act over another when he is indecisive. This inclination finds it expression in the decision maker's reported predisposition to choose one alternative over another. Minardi and Savochkin formalized this idea using, as primitive, a binary relation, $\succsim$ on the set of ordered pairs of acts. They interpret the relation $(f, g) \succsim (f', g')$ as indicating that the decision maker is more confident that f is at least a good as g than that f' is at least as good as g'. Minardi and Savochkin give necessary and sufficient conditions on $\succsim$ for the existence of a function, μ, assigning to every to pair of Anscombe–Aumann acts, f and g, a real number, $\mu(f, g) \in [0, 1]$ such that $(f, g) \succsim (f', g')$ if and only if $\mu(f, g) \geq \mu(f', g')$. Moreover, under these conditions the function μ is a capacity of the subset of the set of priors according to which the expected utility of f is greater or equal to that of g.

To connect the model of this paper to the work of Minardi and Savochkin (2015), consider the special case of doubleton menus. Let $M = \{f, g\}$ and denote by $\zeta(f, g)$ the probability that f is chosen from the menu M. According to our approach, $\zeta(f, g)$ is given in (6).

If we assume that the ordinal ranking of the elements of X is independent of the decision maker's state of mind (e.g., if X are monetary payoff), it is easy to verify that $\zeta(f, g)$ satisfies the properties of the function μ of Minardi and Savochkin (2015).[12] Thus, the application of our approach to doubleton menus of Anscombe–Aumann acts whose payoffs are roulette lotteries over monetary outcomes yields a result which is analogous to that of Minardi and Savochkin.

Unlike Minardi and Savochkin (2015), whose concern is incomplete beliefs, in our model, there is a single prior on the mental state space, a set of material state-dependent utility functions on outcomes and probability measures on the material state space representing the decision maker's states of the mind. In our model, distinct states of mind may represent distinct tastes (e.g., risk attitudes) and/or beliefs on the material state space, corresponding to different moods of the decision maker.

3.3. Random choice behavior

A random choice rule is an assignment of a probability distribution to every feasible set of alternatives, depicting the relative frequencies according to which a decision maker chooses these alternatives. A random utility function is a (finitely additive) probability measure on a set of utility functions. Gul and Pesendorfer (2006) gave necessary and sufficient conditions for a random choice rule to maximize a random utility function when the set of outcomes is finite and the set of utility functions is the von Neumann–Morgenstern utilities over distributions on the set of outcomes.

The model in this paper is close in spirit to the Gul–Pesendorfer representation of random choice rules.[13] What Gul and Pesendorfer call a decision problem is a menu $M = \{p_1, \ldots, p_n\} \subset \Delta(X)$, and what they refer to as a regular random utility function may be restated in terms of our model as follows: Define

$$N^+(M, p) = \{\omega \in \Omega \mid p \succ_\omega p',\ \forall p' \in M, p' \neq p\}.$$

Then η is *regular* if,

$$\eta\left(\cup_{p\in M} N^+(M, p)\right) = 1, \quad \forall M \in \mathcal{M}.$$

If η is regular, then the random choice rule implied by our model assigns p_i, $i = 1, \ldots, n$, the probability

$$\alpha_M(p_i) = \sum_{\omega \in \Psi_M(p_i)} \eta(\omega). \tag{7}$$

By definition, α is a random choice rule represented by the random utility model depicted in Section 2. It can be shown that the

[12] The properties are: Reflexivity (i.e., $\zeta(f, f) = 1$). Weak transitivity (i.e., for all $f, g, h \in H$, $\zeta(f, g) = 1$ implies $\zeta(f, h) \geq \zeta(g, h)$). Monotonicity (i.e., $f_M^*(\omega) = f$ for all $\omega \in \Omega$ implies $\zeta(f, g) = 1$). Independence (i.e., for all $f, g, h \in H$ and $\alpha \in (0, 1]$, $\zeta(f, g) = \zeta(\alpha f + (1-\alpha)h, \alpha g + (1-\alpha)h)$). Reciprocity (i.e., for all $f, g \in H$, $\zeta(f, g) \in (0, 1]$ implies $\zeta(g, f) = 1 - \zeta(f, g)$). Continuity (i.e., for all $f, g, h \in H$ and $\gamma \in [0, 1]$, the sets $\{\alpha \in [0, 1] \mid \eta(\alpha f + (1-\alpha)g, h) \geq \gamma\}$ and $\{\alpha \in [0, 1] \mid \eta(h, \alpha f + (1-\alpha)g) \geq \gamma\}$ are closed). Non-degeneracy (i.e., $\zeta(f, g) = 0$, for some $f, g \in H$). If, in addition, we assume that the ordinal ranking of the elements of X is independent of the decision maker's state of mind, and consider doubleton menus $M = \{\delta_x, \delta_y\}$, then our model implies C-Completeness (i.e., either $\zeta\left(\delta_x, \delta_y\right) = 1$ or $\zeta\left(\delta_y, \delta_x\right) = 1$).

[13] Since Gul and Pesendorfer (2006) and Ahn and Sarver (2013) deal with menu choice under risk, we present the random choice behavior in the same context. However, the discussion should make it clear that the same logic applies to stochastic choice under uncertainty.

random choice rule α satisfies the axioms of Gul and Pesendorfer (2006).[14]

Ahn and Sarver (2013) synthesized the random choice model of Gul and Pesendorfer (2006) and the menu choice model of Dekel et al. (2001) to obtain a representation of a two-stage decision process in which, in the first stage, decision makers choose among menus and their preferences have a representation à la Dekel et al., and in the second stage, they make a stochastic choice from the menu selected in the first stage according to a distribution function (and a tie breaking rule) that has a Gul–Pesendorfer representation. Ahn and Sarver identify the necessary and sufficient conditions for the representation of Dekel et al. (2001) and that of Gul and Pesendorfer (2006) to be consistent, in the sense that the decision maker's prior on the subjective state space and state-dependent utility functions agree with the distribution depicting his stochastic choice behavior and the corresponding state-dependent utility functions of Gul and Pesendorfer.

As explained in Introduction, the model presented in this paper assumes that this synthesis exists. If M is a menu that was selected in the first stage then, as we assumed in Section 3.1, when facing a choice among lotteries, the decision maker anticipates that a state of mind from Ω is drawn according to the distribution η, and that this state of mind determines his choice. If this anticipation is correct then the probability that p_i is chosen is given Eq. (7).

Despite the similarity, the random choice behavior in this paper is fundamentally different from that of Gul and Pesendorfer (2006) and Ahn and Sarver (2013). First, and foremost, in these models the function that associates each menu with a probability distribution over its elements, the random choice rule, is a primitive concept. By contrast, in the model of Section 2, it is a derived concept. Second, the essence of the model of Ahn and Sarver (2013) is consistency between the (ex-ante) anticipated choice and (ex-post) actual stochastic choice, which is exogenously given. The essence of the present model is consistency between the introspective beliefs and the actual beliefs, represented by the probabilities on the mental state space.

Lu (2014) and Dillenberger et al. (2014) address the issue of identifying the distribution of private information signals from choice behavior. Both invoke preference relation on the nonempty subsets of Anscombe and Aumann (1963) acts but take different approaches. Lu (2014) extends the random choice model of Gul and Pesendorfer (2006) to include decision problems that consist of Anscombe–Aumann acts. In the individual interpretation of Lu's model, a decision maker receives a signal that affects his choice behavior. The signal is a draw from a distribution on a canonical signal space of beliefs (that is, prior distribution of the material state space) and tastes (that is, a set of utility functions) and is private information. Lu provides an axiomatic characterization of the random choice rule that is necessary and sufficient for it to have an information representation.[15]

Dillenberger et al. (2014) propose a theory of subjective learning according to which the preference relations on menus of Anscombe and Aumann (1963) acts reflect decision makers' anticipated acquisition of private information before a choice of an act from the menu must be made. They analyze the axiomatic structure that allows an uninformed observer to infer from decision makers' choice behavior, the distribution of the signals (that is, underlying information structure) that govern their ex post choices.[16]

Despite sharing some features with the theory advanced here, the works of Lu (2014) and Dillenberger et al. (2014) are different from the one of this paper conceptually, methodologically, and structurally. To begin with, their objective is a representation of an analyst's inference of the decision maker's private information from his choice behavior. By contrast, in this paper it is the decision maker who is unsure about his own preferences, and the main thrust of our analysis is the representation of the decision maker's beliefs about the evolution of his own preferences and choice behavior. The analysis of Lu and Dillenberger et al. is based on preference relation among menus and are anchored in the revealed preference methodology. By contrast, the model of this paper requires that, in addition to preference relation on menus, the decision maker expresses his preferences on a set of hypothetical lotteries. This departure from the revealed preference methodology has its benefits in terms of its greater generality. Specifically, this work is concerned with the decision maker's uncertainty about his preferences which include beliefs as well as his tastes. Moreover, the information structure, which is focus of the analysis of Dillenberger et al. (2014), corresponds to the beliefs in the model of this paper. Their model and analysis neither intended nor can it address the issue of uncertain tastes which is at the core of the present analysis. Finally, analytical framework, the axiomatic structures and the representations of the preferences in the works of Lu and Dillenberger et al. models are different from those presented here.

4. Concluding remarks

4.1. Menu choice and choice from menus

The use of menu choice in this paper is based on the tacit assumption that choice behavior involves two points in time, the time of choice of a menu, say t_0, and, subsequently, at time t_1, a choice of an element from the menu. The first is not a random choice, the second is. To grasp the difference consider two acts f and f' and the corresponding degenerate menus $M = \{f\}$ and $M' = \{f'\}$. According to Theorem 1, at the choice at time t_0 between M and M' is non-random. Specifically, since $f = f_M$ and $f' = f_{M'}$, by (1) $f \succcurlyeq f'$ if and only if $\sum_{\omega\in\Omega} \eta(\omega)\left[u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega))\right] \geq 0$. However, if at time t_1 the decision maker is presented with a choice between the acts f and f', this is a choice from the menu $\{f, f'\}$. According to our theory, his choice is determined by his state of mind and, therefore, is random. In particular, if his state of mind at time t_1 is ω, he will choose f if $u(\omega, f) > u\left(\omega, f'\right)$ and, disregarding indifference, will choose f', otherwise.

4.2. A remark on methodology

The stochastic choice model advanced in this paper links actual choice among menus to introspective beliefs concerning one's moods and/or possible persona. The representation of introspective beliefs is derived from verbally expressed preferences among

14 If M is such that $\varphi_M(\omega)$ is a singleton for all $\omega \in \Omega$, then, for all $p \in M \subset M'$, $\Psi_M(p) \supset \Psi_{M'}(p)$. Hence, $\alpha^M(p) \geq \alpha^{M'}(p)$. Hence, α is monotone. For all $q \in \Delta(X)$ and $\lambda \in (0, 1]$ let $\hat{M} = \lambda M + (1-\lambda)\{q\} := \{\lambda p_i + (1-\lambda) q \mid p_i \in M\}$. Then, by independence, for all $p \in M$, $\Psi_M(p) = \Psi_{\hat{M}}(\lambda p + (1-\lambda) q)$. Hence, $\alpha^M(p) = \alpha^{\hat{M}}(\lambda p + (1-\lambda) q)$. Thus, α is linear. Also, since in our model, decision makers are expected utility maximizers, restricting choice to extreme points entails no essential loss. So α is extreme. Finally, for all $M, M' \in \mathcal{M}$, and $\lambda \in [0, 1]$ let $\lambda M + (1-\lambda) M' = \{\lambda p + (1-\lambda) p' \mid p \in M, p' \in M'\}$. Then, $\Phi_{\lambda M+(1-\lambda)M'}(\lambda p + (1-\lambda) p') = \{\omega \in \Omega \mid \lambda p + (1-\lambda) p' \in \varphi_{\lambda M+(1-\lambda)M'}(\omega)\}$. But $\lambda p + (1-\lambda) p' \in \varphi_{\lambda M+(1-\lambda)M'}(\omega)$ implies $p \in \varphi_M(\omega)$ and $p' \in \varphi_{M'}(\omega)$. Hence, variations in λ will not change $\Phi_{\lambda M+(1-\lambda)M'}(\lambda p + (1-\lambda) p')$. Thus, α is mixture continuous.

15 Lu (2014) contains additional results and analysis that are not directly related to this work.

16 Dillenberger et al. (2014) also apply their model to the analysis of dynamic decision making tracing the effect of anticipated arrival of information. These aspects of their paper are not directly related to this work.

hypothetically state-acts lotteries. Methodologically speaking, this is not a revealed preference theory. We presume, however, that decision makers are able to express such preferences. Consider, for example, a professor who has just being asked to revise and resubmit a paper for journal publication. Suppose that this request signals that the chances of eventual acceptance increased from 20% to 80%. Let there be two lotteries: Lottery A offers a 60% chance of winning a ticket to performance of Beethoven's 9th symphony and 40% chance of winning a ticket to a performance of Mozart's requiem, and lottery B offers a 40% chance of winning a ticket to performance of Beethoven's 9th symphony and 60% chance of winning a ticket to a performance of Mozart's requiem. Suppose that both concerts are scheduled to take place shortly after the expected final decision on the publication. It is reasonable to suppose that prior to receiving the revise and resubmit decision the professor expresses preferences for lottery A, and after receiving the news he would prefer lottery B. We presume that such expressions are meaningful testimony to his attitudes and that these attitudes are consistent with his actual choice behavior.

But these are preferences on state-act lotteries. Specifically, there are two (constant) acts, namely, attending the performance of Beethoven's 9th symphony and attending the performance of Mozart's requiem. There are two moods, elation associated with the article being accepted and sadness associated with a rejection. Prior to receiving the news, the state-act lotteries corresponding to lotteries A and B, respectively are

Act \ Mood	Elation	Sadness
Beethoven's 9th	0.12	0.48
Mozart's requiem	0.08	0.32

and

Act \ Mood	Elation	Sadness
Beethoven's 9th	0.08	0.32
Mozart's requiem	0.12	0.48

The corresponding lotteries after receiving the news are

Act \ Mood	Elation	Sadness
Beethoven's 9th	0.48	0.12
Mozart's requiem	0.32	0.08

and

Act \ Mood	Elation	Sadness
Beethoven's 9th	0.32	0.08
Mozart's requiem	0.48	0.12

Hence, the aforementioned expression of preferences are preferences on these state-acts lotteries.

Direct verbal interrogation is regarded with suspicion by decision theorists. Savage (1972) put it very bluntly: "If the state of mind in question is not capable of manifesting itself in some sort of extraverbal behavior, it is extraneous to our main interest..." (Savage, 1972, p. 28). Yet, in another passage, Savage proposes an approach which he refers to as mode of interrogation between behavioral and direct. "One can, namely, ask the person, not how he feels but what he would do in such and such situation. In so far as the theory of decision under development is regarded as empirical one, the intermediate mode is a compromise between economy and rigor. But in the theory's more normative interpretation as a set of criteria of consistency for us to apply to our decisions, the intermediate mode is just the right one." (Savage, 1972, p. 28). The intermediate mode alluded to by Savage can be applied to elicit the utility functions using the expressed preferences among state-acts lotteries.

One advantage of our model is that it allows for a more structured analysis of the response of the decision maker to signals that make him update his beliefs about the mental states (e.g. according to Bayes rule). The editorial decision discussed above or, a good (bad) medical report generating a sense of optimism (pessimism) that affect the likelihoods of distinct states of mind, are but a couple of concrete examples. Such signals, result in corresponding changes in the decision maker's preferences over menus and her random choice. In the literatures dealing with menu choice and random choice, such changes are exogenous changes in tastes and/or beliefs. In our model, the same changes are response to belief updating and therefore, are predictable. Similarly, using our model it would be possible to predict the effects of "mood altering drugs" (e.g., anti-depressants) that affect the likelihoods of moods (e.g., first-order stochastic shift towards more upbeat moods) on random choice behavior.

The model presented here presumes that the set of mental states are known to outside observers (e.g., econometricians). Consequently, if the choices from menus correspond to the underlying mental states then it would be possible, using revealed preference methods, to assess the empirical distribution of the mental states from the relative frequencies of observed choices.[17] We hypothesize that the decision maker beliefs about the likely realizations of his mental states, quantified by the probability distribution η, are consistent with the ex post empirical distribution. In this context, we note that the existing probability elicitation procedures are based on the odds decision makers' are willing to accept when betting on events in the (material) states space. These methods are not applicable when the state space under consideration is the mental state space because, unlike the material state space which is ex post public information, the mental states are private information so betting on events in this state space is meaningless.

Finally, it is worth noting that in a recent paper, Karni (2015a) proposes a new mechanism designed to elicit the range of the set of priors and, at the same time, the decision maker's introspective beliefs about the likelihoods of the truth of the priors in the set. If states of mind correspond to distinct beliefs rather than tastes, this mechanism lends revealed preference interpretation of decision makers' introspective beliefs about the likelihoods of the truth of their prior beliefs.

5. Proofs

5.1. Proof of Lemma 1

Let $f_M, f_{M'} \in F$ and $\alpha \in [0, 1]$. By definition $f_M(\omega) \succcurlyeq_\omega h, \forall h \in M$ and $f_{M'}(\omega) \succcurlyeq_\omega h, \forall h \in M'$. We need to show that there exist $\hat{M} \in \mathcal{M}$ such that $f_{\hat{M}} = \alpha f_M + (1-\alpha) f_{M'}$. Consider the menu

$$\hat{M} = \{\alpha f_M(\omega) + (1-\alpha) f_{M'}(\omega) \mid \omega \in \Omega\}.$$

Then, by two applications of (A.3), for all $\omega' \in \Omega$

$$\alpha f_M(\omega) + (1-\alpha) f_{M'}(\omega) \succcurlyeq_\omega \alpha f_M\left(\omega'\right) + (1-\alpha) f_{M'}\left(\omega'\right).$$

Hence $\alpha f_M + (1-\alpha) f_{M'} = f_{\hat{M}}$ and is, by definition, an element of F. Thus, F is a convex set. □

5.2. Proof of Lemma 2

Let $\ell, \ell' \in L(\Omega, H)$ and $\alpha \in [0, 1]$. To show that $\alpha\ell + (1-\alpha)\ell' \in L(\Omega, H)$, we need to show that $J\left(\alpha\ell + (1-\alpha)\ell'\right) \in F$. By definition, for all $\omega \in \Omega$

$$J\left(\alpha\ell + (1-\alpha)\ell'\right)(\omega) = \sum_{h\in H}\left[\frac{\ell(\omega,h)}{\mu_\ell(\omega)}\right]\frac{\alpha\mu_\ell(\omega)}{\alpha\mu_\ell(\omega) + (1-\alpha)\mu_{\ell'}(\omega)}h + \sum_{h\in H}\left[\frac{\ell'(\omega,h)}{\mu_{\ell'}(\omega)}\right]\frac{(1-\alpha)\mu_{\ell'}(\omega)}{\alpha\mu_\ell(\omega) + (1-\alpha)\mu_{\ell'}(\omega)}h.$$

[17] This would correspond to the random choice rule of Gul and Pesendorfer (2006).

Let $\frac{\alpha\mu_\ell(\omega)}{\alpha\mu_\ell(\omega)+(1-\alpha)\mu_{\ell'}(\omega)} = \beta$, then $J\left(\alpha\ell + (1-\alpha)\ell'\right)(\omega) = \beta J(\ell)(\omega) + (1-\beta) J\left(\ell'\right)(\omega)$, for all $\omega \in \Omega$. Hence, by the convexity of F, $J\left(\alpha\ell + (1-\alpha)\ell'\right) = \beta J(\ell) + (1-\beta) J\left(\ell'\right) \in F$. □

5.3. Proof of Theorem 1

(a) (Sufficiency) By Lemma 2, $L(\Omega, H)$ is a convex set. Since $\succ^*$ satisfies (A.1)–(A.4), by the expected utility theorem, there exist continuous, non-constant, real-valued function u on $\Omega \times H$ which is affine in its second argument and unique up to positive linear transformation, such that, for all $\ell, \ell' \in L(\Omega, H)$

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{\omega\in\Omega}\sum_{h\in H} u(\omega, h)\,\ell(\omega, h) \geq \sum_{\omega\in\Omega}\sum_{h\in H} u(\omega, h)\,\ell'(\omega, h). \tag{8}$$

Hence, (2) holds. Note that when ℓ is non-degenerate, (8) can be rewritten as

$$\ell \succcurlyeq^* \ell' \Leftrightarrow \sum_{\omega\in\Omega} \mu_\ell(\omega)\, u(\omega, J(\ell)(\omega)) \geq \sum_{\omega\in\Omega} \mu_{\ell'}(\omega)\, u\left(\omega, J\left(\ell'\right)(\omega)\right).$$

Since F is a convex set and $\succcurlyeq$ satisfies (A.1)–(A.4), there exist continuous, non-constant, real-valued function v on $\Omega \times H$, affine in its second argument and unique up to cardinal unit-comparable transformation, such that for all $f_M, f_{M'} \in F$,

$$f_M \succcurlyeq f_{M'} \Leftrightarrow \sum_{\omega\in\Omega} v(\omega, f_M(\omega)) \geq \sum_{\omega\in\Omega} v(\omega, f_{M'}(\omega)). \tag{9}$$

Fix $\bar{h}$, an interior point of H, let $B^{\bar{h}} \subset H$ be a closed ball centered at $\bar{h}$, denote by $\bar{h}(\omega)$ the unique maximizer of $\succcurlyeq_\omega$ over $B^{\bar{h}}$ and let $\bar{M} = \left\{\bar{h}(\omega) \mid \omega \in \Omega\right\}$ be a menu consisting of these maximal points. Since $\omega \neq \omega'$ implies $\succ_\omega \neq \succ_{\omega'}$, $h(\omega) \neq h\left(\omega'\right)$ and each $\bar{h}(\omega)$ has a neighborhood $N_\omega^{\bar{h}}$ such that, for all $h \in N_\omega^{\bar{h}}$ and $\omega' \neq \omega$, $h \succ_\omega \bar{h}\left(\omega'\right)$ and $\bar{h}\left(\omega'\right) \succ_{\omega'} h$. Hence, for all $h \in N_\omega^{\bar{h}}$ and $\bar{M}_\omega = \{h\} \cup \left\{\bar{h}\left(\omega'\right) \mid \omega' \neq \omega,\ \omega' \in \Omega\right\}$,

$$f_{\bar{M}_\omega}\left(\omega'\right) = \begin{cases} h & \omega' = \omega \\ \bar{h}\left(\omega'\right) & \omega' \neq \omega. \end{cases}$$

Define $\ell_{\bar{M}} = K\left(f_{\bar{M}}\right)$ and consider an obviously nonnull $\omega \in \Omega$. Let L_ω denote the subset of lotteries in $L(\Omega, H)$ that agree with $\ell_{\bar{M}}$ outside ω and denote $F_\omega = J(L_\omega)$. By (A.5), $\succcurlyeq^*$ restricted to lotteries in L_ω and $\succcurlyeq$ restricted to F_ω agree (that is, for all $\ell, \ell' \in L_\omega$, and $J(\ell), J\left(\ell'\right) \in F_\omega$ then $\ell \succcurlyeq^* \ell'$ if and only if $J(\ell) \succcurlyeq J\left(\ell'\right)$).[18] For the restricted relations $\succcurlyeq^*$ and $\succcurlyeq$ the functions $u(\omega, \cdot)$ and $v(\omega, \cdot)$ constitute, respectively, von Neumann–Morgenstern utility functions on the subset of acts $H_\omega = \{h \in H \mid h = f_M(\omega),\ f_M \in F_\omega\}$. By the preceding argument, H_ω contains the open neighborhood $N_\omega^{\bar{h}}$. Hence, by the affinity and uniqueness of the von Neumann–Morgenstern utility functions $u(\omega, \cdot)$ and $v(\omega, \cdot)$, $v(\omega, \cdot) = b(\omega)\, u(\omega, \cdot) + a(\omega)$, where $b(\omega) > 0$. As each $v(\omega, \cdot)$ can be rescaled by subtracting $a(\omega)$, we assume (without loss of generality) that $v(\omega, \cdot) = b(\omega)\, u(\omega, \cdot)$. For ω obviously null let $b(\omega) = 0$. Thus, $v(\omega, h) = b(\omega)\, u(\omega, h)$, for all $\omega \in \Omega$ and $h \in H$.

By (A.4), there exist an obviously nonnull state of mind, thus, $b(\omega) > 0$ for some $\omega \in \Omega$. Define $\eta(\omega) = b(\omega) / \Sigma_{\omega'\in\Omega} b\left(\omega'\right)$ and observe that, by (9),

$$f_M \succcurlyeq f_{M'} \Leftrightarrow \sum_{\omega\in\Omega} \eta(\omega)\left[u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega))\right] \geq 0. \tag{10}$$

Hence, (1) holds.

(Necessity) The proof is immediate and is omitted.

(b) This is an immediate implication of the uniqueness of u in (8).

(c) Assume that all ω are obviously nonnull and consider the menu $\bar{M}$ defined in part (a). Since ω is obviously nonnull then there are $f, f' \in F$, such that f agrees with f' outside ω, and $f \succ f'$. Hence, (1) implies that $\eta(\omega)\left[u(\omega, f(\omega)) - u\left(\omega, f'(\omega)\right)\right] > 0$. Thus, $\eta(\omega) > 0$ for all ω.

Suppose that there exists $\eta' \neq \eta$ that, in conjunction with u, satisfy the representations in (1) and (2). We may write (1) as

$$\begin{aligned} J(\ell) &\succcurlyeq J\left(\ell'\right) \\ &\Longleftrightarrow \sum_{\omega\in\Omega} \eta(\omega)\left[u(\omega, J(\ell)(\omega)) - u\left(\omega, J\left(\ell'\right)(\omega)\right)\right] \geq 0 \\ &\Longleftrightarrow \sum_{\omega\in\Omega} \eta'(\omega)\left[u(\omega, J(\ell)(\omega)) - u\left(\omega, J\left(\ell'\right)(\omega)\right)\right] \geq 0. \end{aligned}$$

Since $\eta \neq \eta'$, there are $\omega, \omega' \in \Omega$, such that $\eta(\omega) > \eta'(\omega)$ and $\eta\left(\omega'\right) < \eta'\left(\omega'\right)$. For $p \in [0, 1]$ define

$$\ell_p\left(\omega, \bar{h}(\omega)\right) = \eta(\omega)\, p, \qquad \ell'_p\left(\omega', \bar{h}\left(\omega'\right)\right) = \eta\left(\omega'\right)(1-p)$$
$$\ell_p\left(\omega, \bar{h}\left(\omega'\right)\right) = \eta(\omega)(1-p), \qquad \ell'_p\left(\omega', \bar{h}(\omega)\right) = \eta\left(\omega'\right) p$$
$$\ell_p\left(\omega', \bar{h}(\omega)\right) = \eta\left(\omega'\right) \quad \text{and} \quad \ell'_p\left(\omega, \bar{h}\left(\omega'\right)\right) = \eta(\omega)$$

and, for all $\omega'' \in \Omega \setminus \{\omega, \omega'\}$, $\ell_p\left(\omega'', \bar{h}\left(\omega''\right)\right) = \ell'_p\left(\omega'', \bar{h}\left(\omega''\right)\right)$. Then

$$J\left(\ell_p\right)(\omega) = p\bar{h}(\omega) + (1-p)\,\bar{h}\left(\omega'\right), J\left(\ell_p\right)\left(\omega'\right) = \bar{h}(\omega),$$

and

$$J\left(\ell'_p\right)\left(\omega'\right) = (1-p)\,\bar{h}\left(\omega'\right) + p\bar{h}(\omega), J\left(\ell'_p\right)(\omega) = \bar{h}\left(\omega'\right)$$

and $J\left(\ell_p\right)\left(\omega''\right) = J\left(\ell'_p\right)\left(\omega''\right)$, for all $\omega'' \in \Omega \setminus \{\omega, \omega'\}$.

By definition, $\bar{h}(\omega) \succ_\omega \bar{h}\left(\omega'\right)$ and $\bar{h}\left(\omega'\right) \succ_{\omega'} \bar{h}(\omega)$. Hence, $J\left(\ell_p\right) \succcurlyeq J\left(\ell'_p\right)$ if and only if

$$p\eta(\omega)\left[u\left(\omega, \bar{h}(\omega)\right) - u\left(\omega, \bar{h}\left(\omega'\right)\right)\right] + (1-p)\,\eta\left(\omega'\right)\left[u\left(\omega', \bar{h}(\omega)\right) - u\left(\omega', \bar{h}\left(\omega'\right)\right)\right] \geq 0$$

if and only if

$$p\eta'(\omega)\left[u\left(\omega, \bar{h}(\omega)\right) - u\left(\omega, \bar{h}\left(\omega'\right)\right)\right] + (1-p)\,\eta'\left(\omega'\right)\left[u\left(\omega', \bar{h}(\omega)\right) - u\left(\omega', \bar{h}\left(\omega'\right)\right)\right] \geq 0.$$

But $u\left(\omega, \bar{h}(\omega)\right) - u\left(\omega, \bar{h}\left(\omega'\right)\right) > 0$ and $u\left(\omega', \bar{h}(\omega)\right) - u\left(\omega', \bar{h}\left(\omega'\right)\right) < 0$. Moreover, $\eta(\omega) > 0$ and $\eta'\left(\omega'\right) > 0$. Thus, there exists $\bar{p}$ such that

$$\bar{p}\eta(\omega)\left[u\left(\omega, \bar{h}(\omega)\right) - u\left(\omega, \bar{h}\left(\omega'\right)\right)\right] + (1-\bar{p})\,\eta\left(\omega'\right)\left[u\left(\omega', \bar{h}(\omega)\right) - u\left(\omega', \bar{h}\left(\omega'\right)\right)\right] = 0$$

but then, since $\eta(\omega) > \eta'(\omega)$ and $\eta\left(\omega'\right) < \eta'\left(\omega'\right)$

$$\bar{p}\eta'(\omega)\left[u\left(\omega, \bar{h}(\omega)\right) - u\left(\omega, \bar{h}\left(\omega'\right)\right)\right] + (1-\bar{p})\,\eta'\left(\omega'\right)\left[u\left(\omega', \bar{h}(\omega)\right) - u\left(\omega', \bar{h}\left(\omega'\right)\right)\right] < 0.$$

This is the required contradiction. □

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[18] See Karni and Schmeidler (2016).

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